FORCED OSCILLATIONS IN A HEAT- AND MASS-TRANSFER SYSTEM WITH A FINITE TRANSFER RATE

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UDC 536.24:541.182

The solution of the general boundary-value problem for the one-dimensional inhomogeneous heat- and mass-transfer equation with a finite transfer rate is investigated. The forced oscillations are analyzed in several special cases.

According to the theory developed by Lykov [1], certain processes in heat- and mass-transfer systems are significantly affected by the presence of a finite transfer rate. For example, the propagation of heat and mass in heat- and moisture-transfer processes in capillary-porous bodies takes place with a finite velocity. The significant factor here is that the allowance for a finite transfer rate leads to hyperbolic-type equations. The moisture conductivity equation has the form

$$\frac{\partial^2 u}{\partial \tau^2} + \frac{w_r^2}{a_m} \cdot \frac{\partial u}{\partial \tau} = w_r^2 \Delta u, \tag{1}$$

in which $w_r^2 = a_m/\tau_r$ is the capillary velocity of the liquid, τ_r is the relaxation period in hours, and a_m is the moisture diffusivity.

Certain problems associated with Eq. (1) have been solved by Lykov and Perel'man, as well as by other authors [2, 3].

We propose to analyze the general boundary-value problem for the one-dimensional inhomogeneous equation

$$b \frac{\partial^2 u}{\partial \tau^2} + c \frac{\partial u}{\partial \tau} = a \frac{\partial^2 u}{\partial x^2} + f(x, \tau) \tag{2}$$

subject to the initial conditions

$$u(x, 0) = \psi_1(x), \quad u_\tau(x, 0) = \psi_2(x)$$

and boundary conditions

$$\begin{aligned} \alpha_1 u_x(x_1, \tau) + \beta_1 u(x_1, \tau) &= \varphi_1(\tau), \\ \alpha_2 u_x(x_2, \tau) + \beta_2 u(x_2, \tau) &= \varphi_2(\tau), \end{aligned} \tag{3}$$

for $x_1 \le x \le x_2$, $\tau \ge 0$; a, b, c are nonnegative constants and α_1, α_2, β_1, β_2 are constants.

To solve the problem we use the integral transform

$$\bar{g}_j(\tau) = \int_{x_1}^{x_2} K_j(x) g(x, \tau)\, dx, \tag{4}$$

$$g(x, \tau) = \sum_j \bar{g}_j(\tau) K_j(x). \tag{5}$$

The solution is then represented in the form

$$u(x, \tau) = \sum_j \bar{u}_j(\tau) K_j(x). \tag{6}$$

State University, Voronezh. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 21, No. 1, pp. 165-167, July, 1971. Original article submitted July 20, 1970.

The kernel of the transform for the given problem is written in the form

$$K_j(x) = \frac{A_j^{(1)}}{\beta_1 S_j^{(1)} + \alpha_1\mu_j C_j^{(1)}} [(\beta_1 S_j^{(1)} + \alpha_1\mu_j C_j^{(1)}) \cos \mu_j x + (\alpha_1\mu_j S_j^{(1)} - \beta_1 C_j^{(1)}) \sin \mu_j x], \quad (7)$$

where

$$C_j^{(i)} = \cos \mu_j x_i; \; S_j^{(i)} = \sin \mu_j x_i, \; i = 1,2;$$

$$A_j^{(1)^2} = 2\mu_j (\alpha_1\mu_j C_j^{(1)} + \beta_1 S_j^{(1)})^2 \{(\alpha_1\mu_j C_j^{(1)} + \beta_1 S_j^{(1)})^2[\mu_j (x_2 - x_1) + S_j^{(2)}C_j^{(2)} - S_j^{(1)}C_j^{(1)}] + (\alpha_1\mu_j S_j^{(1)} - \beta_1 C_j^{(1)})^2[\mu_j (x_2 - x_1) + C_j^{(1)}S_j^{(1)} - C_j^{(2)}S_j^{(2)}] + (\beta_1 S_j^{(1)} + \alpha_1\mu_j C_j^{(1)})(\alpha_1\mu_j S_j^{(1)} - \beta_1 C_j^{(1)})(C_j^{(1)^2} - C_j^{(2)^2} - S_j^{(1)^2} + S_j^{(2)^2})\}^{-1}. \quad (8)$$

Here the eigenvalues μ_j^2 satisfy the equation

$$[\beta_1 C_j^{(1)} - \alpha_1\mu_j S_j^{(1)}][\beta_2 S_j^{(2)} + \alpha_2\mu_j C_j^{(2)}] = [\beta_2 C_j^{(2)} - \alpha_2\mu_j S_j^{(2)}][\beta_1 S_j^{(1)} + \alpha_1\mu_j C_j^{(1)}]. \quad (9)$$

Now the solution (6) leads to the following equation in the image domain:

$$b\, \bar{u}_j''(\tau) + c\bar{u}_j'(\tau) + a\mu_j^2 \bar{u}_j(\tau) = aB_j(\tau) + \bar{f}_j(\tau), \quad (10)$$

where

$$B_j(\tau) = \frac{A_j^{(1)}}{\alpha_2} \{[\alpha_1\mu_j \cos \mu_j (x_2 - x_1) - \beta_1 \sin \mu_j (x_2 - x_1)]\, \varphi_2(\tau) - \alpha_2\mu_j \varphi_1(\tau)\}$$

subject to the initial condition

$$\bar{u}_j(0) = \bar{\psi}_{1j}, \quad \bar{u}_j'(0) = \bar{\psi}_{2j}.$$

Equation (10) clearly describes forced oscillations excited in the system by external effects specified by the functions

$$f(x, \tau), \; \varphi_1(\tau), \; \varphi_2(\tau).$$

Let us consider a special case of forced oscillations. Let $\alpha_1 = \alpha_2 = 0$, $\beta_1 = \beta_2 = 1$, $\varphi_1 = \varphi_2 = \psi_1 = \psi_2 = 0$, $x_1 = 0$, $x_2 = l$, $f = A \sin \omega\tau$. In this case $\mu_k = k\pi/l$, $K_k(x) = (2/l)\sin(k\pi x/l)$,

$$\bar{f}_k = \frac{2}{l} \int_0^l A \sin \omega\tau \sin \frac{k\pi x}{l}\, dx = \frac{2A}{k\pi} [1 - (-1)^k] \sin \omega\tau = f_k \sin \omega\tau.$$

Now the solution of Eq. (10) can be represented in the final form

$$\bar{u}_k(\tau) = \frac{(a\mu_k^2 - b\omega^2) f_k}{(a\mu_k^2 - b\omega^2)^2 + c^2\omega^2} \sin \omega\tau + \frac{c\omega f_k}{(a\mu_k^2 - b\omega^2)^2 + c^2\omega^2} \cos \omega\tau. \quad (11)$$

Substituting (11) and the kernel $K_k(x)$ into (6), we obtain a solution of the forced oscillation problem.

In the theory of thermal conduction the velocity of heat propagation is infinite, the relaxation period $\tau_r = 0$, and we arrive at a parabolic-type equation, putting $b = 0$ in (2). Then (11) assumes the form

$$\bar{u}_k(\tau) = \frac{a\mu_k^2 f_k}{a^2\mu_k^4 + c^2\omega^2} \sin \omega\tau + \frac{c\omega f_k}{a^2\mu_k^4 + c^2\omega^2} \cos \omega\tau. \quad (12)$$

As μ_k^2 is increased the amplitudes of the harmonics decrease, and the solution is determined by the frequency ω.

The periodic solutions of the thermal conduction equation have been investigated in a more general case in [4]. In the case $c = 0$ the equation is of the hyperbolic type, and expression (11) assumes the form

$$\bar{u}_k(\tau) = \frac{f_k}{a\mu_k^2 - b\omega^2} \sin \omega\tau. \quad (13)$$

Clearly, the effect known as resonance can occur here when $(a\mu_k^2 - b\omega^2) \to 0$ for one of the values of k. Then the amplitude of the corresponding harmonic increases without bound, as does the solution of problem (6). This effect can be observed for gases at low pressures, in which case the heat transfer is molecular and the thermal conductivity is determined by the finite velocity of heat propagation.

Finally, setting the denominator in (11) equal to zero, we can establish a more complex relationship between the forced oscillation frequency ω and the eigenvalues μ_k.

LITERATURE CITED

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